

Non-linear perturbations in scalar-tensor cosmologies

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Can local fluctuations of a “Quintessence” scalar field play a dynamical role in the gravitational clustering and cosmic structure formation process? We address this question in the general framework of scalar-tensor theories of gravity. Non-linear energy density perturbations, both in the scalar field and matter component, and linear metric perturbations are accounted for in the perturbed Einstein’s equations. We derive the Newtonian limit of the relevant equations for clustering in scalar-tensor cosmologies. We then specialize to non-linear perturbations of the “Extended Quintessence” model of Dark Energy; in such a model, a non-minimally coupled scalar field is thought to be responsible for driving the present accelerated phase of the Universe expansion. The interplay between Dark Energy and Dark Matter is displayed in the equations governing the growth of structure in the Universe.

I. INTRODUCTION

Our traditional picture of the Universe has been definitely upset when, in 1998, astronomers found that distant type 1a Supernovae were dimmer than expected in a decelerating Universe [1], [2]. This early evidence has been confirmed by the subsequent studies, which combined the most recent Cosmic Microwave Background (CMB) data from WMAP (see [3], [4] and references therein) and Large Scale Structure (LSS) data [5], [6, 7], together with measurements of the Hubble constant [8]; the fact that the Universe turns out to be geometrically close to flat, together with the estimates of its matter content, has called for deep changes of the old “standard” scenario of a matter dominated Universe. According to these observations, almost 70% of the total energy of the Universe resides in a “Dark Energy” component, which is plausibly acting as a repulsive force driving the cosmic acceleration. Although the Cosmological Constant has been historically proposed as a candidate for the cosmic acceleration, the theoretical difficulties in justifying its exceedingly small value motivated the search for alternative theories.

There are currently two different general approaches to Dark Energy modeling; one class of models is based on the introduction of a new cosmological component having negative equation of state, often described through the dynamics of a self-interacting, minimally coupled scalar field [9]–[19]. The other class of models proposes modifications to the gravity itself, introducing a non-minimally coupled scalar field (scalar-tensor theories) or changing the function of the Ricci scalar appearing in the Gravitational sector of the Lagrangian of the theory [20]–[33]. In both pictures, an open important issue is whether modifications of gravity, or small-scale perturbations of a Quintessence scalar field, could lead to significant effects on the formation of structures, such as galaxies and clusters (see [23], [29], [34]–[42]).

A powerful tool for the investigation of the Dark Energy main properties and parameters has been the linear perturbation theory, which allowed to make accurate predictions on many cosmological observables, such as the CMB spectrum of anisotropies (see, e.g., [43]–[47]); in this paper, we deal with the behavior of perturbations in the non-linear regime. We focus on scalar-tensor theories, where a scalar field non-minimally coupled to the Ricci scalar is proposed as a candidate for the Dark Energy component; in principle, the “Extended Quintessence” field [21] allows for non-vanishing small-scale perturbations, which have been analyzed in the linear regime in [21], [27], [29], [48]. In particular, in [48] and [29] it was shown that, while perturbations in a minimally-coupled scalar field behave as radiation on sub-horizon scales [49], so that the field rapidly becomes a smooth component on such scales, perturbations in a non-minimally coupled scalar field can be dragged by perturbations in the matter component, thanks to the coupling of the scalar field to the Ricci scalar, and become non-linear. In practice, the mechanism which would damp out the field perturbations is here counterbalanced by the presence of a source term, which is directly related

to the gravitational potentials and, ultimately, to the matter perturbations: we expect this dragging to be able to affect gravitational clustering down to galactic scales.

We extend here the analysis performed in [29] for the linear perturbations in scalar-tensor theories, obtaining the non-linear Newtonian limit of Einstein's equations for this class of models. These will be the equations to be eventually inserted in a modified N-body code, in order to simulate the behavior of collapsing matter under this modified property of gravity. Some numerical simulations, involving Dark Energy in several context, have been performed in Refs. [50]-[57].

Finally, we focus on the Extended Quintessence model, specializing the Poisson equation for the gravitational potential in order to evaluate modifications with respect to the “standard” theory.

The plan of the paper is as follows: in section II, we write the perturbed Einstein's equations and the equation for scalar field perturbations in non-linear regime, assuming that the metric perturbations are linear, while those of matter and scalar field are not. In sect. III we discuss the approximations giving the Newtonian limit of these equations, and in sect. IV we specify the general equations for scalar-tensor theories to the “Extended Quintessence” case. Finally, in sect. V we draw our conclusions.

II. PERTURBED EINSTEIN EQUATIONS IN EXTENDED QUINTESSENCE

Since we are interested in the problem of how Dark Energy perturbations could affect the structure formation process, excluding the limit of very strong gravitational fields (typical, for example, of black holes), we will consider linear perturbations of the metric tensor, while keeping non-linearity in the matter and scalar field perturbations [40]. This also means that we are restricting our study to non-relativistic Dark Matter, whose typical velocity is much smaller than the speed of light even in the presence of highly non-linear matter overdensities.

Metric perturbations can be decomposed into scalar modes, as well as by vector and tensor ones; by taking the spatial covariant divergence of the $(0-i)$ Einstein equations and the trace of the $(i-j)$ equations, we will single out scalar modes, so that, without loss of generality, the line-element can be written, in conformal Newtonian-gauge formalism, in terms of only two gravitational potentials Φ and Ψ as

$$ds^2 = a^2(\eta)[-(1 + 2\Phi)d\eta^2 + (1 - 2\Psi)\delta_i^j dx^i dx^j] = g_{\mu\nu} dx^\mu dx^\nu, \quad (1)$$

where η denotes conformal time; note that we adopt the signature $(-+++)$. The Lagrangian for scalar-tensor theories reads

$$\mathcal{L} = \frac{1}{2\kappa} f(\phi, R) - \frac{1}{2} \omega(\phi) \phi_{,c} \phi^{,c} - V(\phi) + \mathcal{L}_{matter} \quad (2)$$

where $\kappa \equiv 8\pi G^*$ and G^* is the bare gravitational constant.

We will consider a Universe filled by dark matter and a non-minimally coupled scalar field; the matter component will be described in terms a discrete set of particles with constant mass m and coordinates $\mathbf{x}_a(\eta)$ ($a = 1, 2, \dots$). Following [58], if we denote the matter four-velocity by $u^\mu \equiv dx^\mu/dx^0$ ($x^0 \equiv \eta$), the matter stress-energy tensor reads

$$T_\nu^\mu \sim a^{-2} \rho_m u^\mu u_\nu \quad (3)$$

where we neglected corrections $\mathcal{O}|\mathbf{u}|^2$, and we defined

$$\rho_m = m a^{-3} \sum_a \delta^{(3)}(\mathbf{x} - \mathbf{x}_a) \quad (4)$$

The scalar field stress-energy tensor reads

$$T_\beta^\alpha = \omega \left[g^{\alpha\nu} \phi_{,\nu} \phi_{,\beta} - \frac{1}{2} g^{\alpha\nu} g_{\nu\beta} \phi_{,\sigma} \phi^{,\sigma} \right] - g^{\alpha\nu} g_{\nu\beta} V + \frac{1}{2} g^{\alpha\nu} g_{\nu\beta} \left(\frac{f}{\kappa} - R F \right) + g^{\alpha\nu} F_{,\nu;\beta} - g^{\alpha\nu} g_{\nu\beta} F^{,\sigma}_{;\sigma} + \left(\frac{1}{\kappa} - F \right) G_\beta^\alpha \quad (5)$$

where, R is the Ricci scalar and $F \equiv \frac{1}{\kappa} \frac{\partial f}{\partial R}$. Restricting ourselves to the simplest case of $f(\phi, R)$ linear in R , (as is the case of the “Extended Quintessence” model), it turns out that $FR = \frac{f(\phi, R)}{\kappa}$. We also assume that $\omega = \text{const} = 1$, so that the tensor (5) becomes

$$T_{\alpha\beta} = \phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}g_{\alpha\beta}\dot{\phi}\dot{\phi} - Vg_{\alpha\beta} + F_{,\alpha;\beta} - g_{\alpha\beta}F^{,\sigma}{}_{;\sigma} + \left(\frac{1}{\kappa} - F\right)G_{\alpha\beta} \quad (6)$$

Both the scalar field and the function $F(\phi(\mathbf{x}, \eta))$ can be written as the superposition of a background component, only dependent on time, plus a generally non-linear perturbation, which depends both on the spatial coordinates and on time:

$$\phi(\mathbf{x}, \eta) \equiv \phi_b(\eta) + \delta\phi(\mathbf{x}, \eta) \quad F(\phi(\mathbf{x}, \eta)) \equiv F_b(\eta) + \delta F(\mathbf{x}, \eta) \quad , \quad (7)$$

The evolution of the background quantities is fully determined by the Friedmann equations and the unperturbed Klein-Gordon equation, which read, respectively,

$$\mathcal{H}^2 = \frac{1}{3F_b} \left(a^2 \langle \rho_m \rangle + \frac{1}{2} \dot{\phi}_b^2 + a^2 \langle V \rangle - 3\mathcal{H}\dot{F}_b \right) \quad ; \quad (8)$$

$$\dot{\mathcal{H}} = -\frac{1}{6F_b} \left(a^2 \langle \rho_m \rangle + 2\dot{\phi}_b^2 - 2a^2 \langle V \rangle + 3\ddot{F}_b \right) \quad , \quad (9)$$

where $\langle V \rangle \equiv V(\phi_b(\eta))$ and $\langle \rho_m \rangle$ is the background value of the matter energy-density. Finally, the background Klein-Gordon equation in scalar-tensor theories reads

$$\ddot{\phi}_b + 2\mathcal{H}\dot{\phi}_b - \frac{1}{2}a^2 R_b F'_b + a^2 V'(\phi_b) = 0 \quad , \quad (10)$$

where a prime denotes differentiation with respect to ϕ .

In our approach to the perturbed Einstein's equations, the Ricci and Einstein's tensors will be linearly perturbed, while the perturbations in the components of the stress-energy tensors (3) and (6) are obtained by subtracting their background (mean) values from the fully non-linear ones:

$$\delta G^\mu_\nu = \kappa (T^\mu_\nu - \langle T^\mu_\nu \rangle) \quad . \quad (11)$$

As for the (0-0) component (“energy constraint”), we get the following perturbation equation:

$$\begin{aligned} \nabla^2 \Phi + 3\mathcal{H}(\dot{\Phi} + \dot{\Psi}) + 3\ddot{\Psi} &= 8\pi G^* \left[\frac{1}{2} \left(\sum_a a^{-1} m_a \delta_{(a)}^{(3)} - a^2 \langle \rho_m \rangle \right) + \sum_a a^{-1} m_a \delta_{(a)}^{(3)} \Phi + \right. \\ &+ (\delta\dot{\phi})^2 + 2\dot{\phi}_b \delta\dot{\phi} - a^2 (V - \langle V \rangle) - 2a^2 \Phi V + \\ &+ \frac{3}{2} \delta\ddot{F} - \frac{3}{2} \dot{F}(\dot{\Phi} + \dot{\Psi}) + \frac{1}{2} \nabla\delta F \cdot (\nabla\Psi - 3\nabla\Phi) - \frac{1}{2} \nabla^2 \delta F (1 + 2\Phi + 2\Psi) + \\ &\left. + 3\mathcal{H}\delta F + \left(\frac{1}{\kappa} - F \right) (3\mathcal{H}\dot{\Psi} + 3\ddot{\Psi} + 3\mathcal{H}\dot{\Phi} + \nabla^2 \Phi) \right] \quad , \quad (12) \end{aligned}$$

where $\delta_{(a)}^{(3)} \equiv \delta^{(3)}(\mathbf{x} - \mathbf{x}_a)$.

Similarly, the equation for the divergence of the (0- i) or “momentum constraint” equation (herefrom latin indices will be used to label spatial coordinates) reads

$$\begin{aligned} \mathcal{H} \nabla^2 \Phi + \nabla^2 \dot{\Psi} &= 4\pi G^* \left[- \sum_a m_a a^{-1} \left(\delta_{(a)}^{(3)} u_{(a)}^i \right)_{,i} + 2\dot{\phi} \nabla\Psi \cdot \nabla\delta\phi + (1 + 2\Psi)\dot{\phi} \nabla^2 \delta\phi + (1 + 2\Psi) \nabla\delta\phi \cdot \nabla\delta\dot{\phi} + \right. \\ &+ \nabla\delta F \cdot (2\mathcal{H}\nabla\Psi - \nabla\dot{\Psi}) + \nabla^2 \delta F (-\mathcal{H} - 2\mathcal{H}\Psi + \dot{\Psi}) + \nabla\delta\dot{F} \cdot (2\nabla\Psi - \nabla\Phi) + \nabla^2 \delta\dot{F} (1 + 2\Psi) - \dot{F} \nabla^2 \Phi - \\ &\left. - 2\nabla\delta F \cdot (\mathcal{H}\nabla\Phi + \nabla\dot{\Psi}) + 2 \left(\frac{1}{\kappa} - F \right) (\mathcal{H}\nabla^2 \Phi + \nabla^2 \dot{\Psi}) \right] \quad (13) \end{aligned}$$

Finally, the trace of the $(i-j)$ component is:

$$\begin{aligned}
& -2\dot{\mathcal{H}}\Phi - \mathcal{H}(\dot{\Phi} + 5\dot{\Psi}) - 4\mathcal{H}^2\Phi - \ddot{\Psi} + \frac{4}{3}\nabla^2\Psi - \frac{1}{3}\nabla^2\Phi = \\
& = 8\pi G^* \left[\frac{1}{2} \left(\sum_a m a^{-1} \delta_{(a)}^{(3)} - a^2 \langle \rho_m \rangle \right) + \frac{1}{3} (1 + 2\Psi) |\nabla \delta\phi|^2 + a^2 (V - \langle V \rangle) + \right. \\
& + \Phi \ddot{F}|_b - \frac{1}{2} (1 - 2\Phi) \delta \ddot{F} - 2\mathcal{H} \delta \dot{F} + \dot{F} (4\mathcal{H}\Phi + \frac{1}{2}\dot{\Phi} + \frac{5}{2}\dot{\Psi}) + \nabla \delta F \cdot \left(-\frac{5}{6}\nabla\Psi + \frac{1}{2}\nabla\Phi \right) + \frac{5}{6}\nabla^2 \delta F (1 + 2\Psi) + \\
& \left. + \left(\frac{1}{\kappa} - F \right) \left(-2\Phi(\dot{\mathcal{H}} + 2\mathcal{H}^2) - \mathcal{H}\dot{\Phi} - \frac{1}{3}\nabla^2\Phi - 5\mathcal{H}\dot{\Psi} - \ddot{\Psi} + \frac{4}{3}\nabla^2\Psi \right) - \delta F(\dot{\mathcal{H}} + 2\mathcal{H}^2) \right] \quad (14)
\end{aligned}$$

The perturbed Klein-Gordon equation gives

$$\begin{aligned}
& \delta \ddot{\phi} + 2\mathcal{H} \delta \dot{\phi} - (\dot{\Phi} + 3\dot{\Psi})(\dot{\phi}_b + \delta \dot{\phi}) - \nabla(\Phi - \Psi) \cdot \nabla \delta \phi - (1 + 2\Phi + 2\Psi) \nabla^2 \delta \phi + a^2 [V'(\phi(\mathbf{x}, \eta)) - V'(\phi_b)] + \\
& 2 a^2 \Phi V'(\phi(\mathbf{x}, \eta)) - \frac{a^2}{2} \delta F' R_b + (F'_b + \delta F')(3\mathcal{H}\dot{\Phi} + \nabla^2\Phi + 3\ddot{\Psi} + 9\mathcal{H}\dot{\Psi} - 2\nabla^2\Psi) = 0 \quad (15)
\end{aligned}$$

where $R_b = 6a^{-2}(\dot{\mathcal{H}} + \mathcal{H}^2)$ is the Ricci scalar in the unperturbed metric. Finally, the equation describing the motion of Dark Matter particles, in the weak-field limit, is given by

$$\ddot{x}_a^i + \dot{x}_a^i \left[\mathcal{H} - \dot{\Phi} - 2\dot{\Psi} \right] = -\partial^i \Phi \quad (16)$$

The system (12)-(16) is a closed system of equations in the variables Φ, Ψ, ϕ, x_a^i 's, which is redundant in the number of equations, as a consequence of the underlying gauge-invariance of Einstein's theory; as we will see, it will be more convenient to disregard the momentum constraint, working with the remaining equations.

All these equations apply to cosmological scales, under the assumption of linear perturbations of the gravitational potentials. In the next section, we will determine the system adequate to determine the matter particle motion on the scales relevant for structure formation, extending the standard Newtonian approximation to the case of scalar-tensor theories.

III. THE EXTENDED NEWTONIAN APPROXIMATION

There are many considerations that can help to simplify the equations (12)-(16). First of all, note that, in the weak-field limit, $|\Phi|$ and $|\Psi|$ are of the order of $|u^i|^2 \ll 1$, $|u^i|$ being the typical velocity of non-relativistic matter; furthermore, since the characteristic evolution time of Ψ and Ψ is $\tau_{dyn} \sim \tau \equiv \mathcal{H}^{-1}$, we can neglect $\dot{\Phi}, \dot{\Psi}$ with respect to \mathcal{H} in the energy constraint equation, in the perturbed Klein-Gordon and in the equation of motion.

Focusing on the perturbation growth on scales well below the Hubble radius, we can apply further approximations to the system. On those scales, $\ddot{\Phi}, \mathcal{H}\dot{\Phi} \sim \mathcal{H}^2\Phi \ll \nabla^2\Phi$ and $\ddot{\Psi}, \mathcal{H}\dot{\Psi} \sim \mathcal{H}^2\Psi \ll \nabla^2\Psi$ and we can neglect such terms in the energy constraint, Einstein spatial trace and Klein-Gordon equations. We also assume that $\dot{\mathcal{H}} \sim \mathcal{H}^2$, so the last term on the right-hand side of the trace equation is negligible, on sub-horizon scales, with respect to the term containing $\nabla^2 \delta F$. Equations (12)-(16) reduce therefore to the following system:

$$\begin{aligned}
\nabla^2 \Phi & = 8\pi G^* \left[\frac{1}{2} \left(\sum_a a^{-1} m_a \delta_{(a)}^{(3)} - a^2 \langle \rho_m \rangle \right) + (\delta\dot{\phi})^2 + 2\dot{\phi}_b \delta\dot{\phi} - a^2 (V - \langle V \rangle) + \right. \\
& + \frac{3}{2} \delta \ddot{F} - \frac{3}{2} \dot{F} (\dot{\Phi} + \dot{\Psi}) - \frac{1}{2} \nabla^2 \delta F + 3\mathcal{H} \delta F + \left(\frac{1}{\kappa} - F \right) \nabla^2 \Phi \left. \right] \quad ; \quad (17)
\end{aligned}$$

$$\begin{aligned} \mathcal{H} \nabla^2 \Phi + \nabla^2 \dot{\Psi} = 4\pi G^* \left[- \sum_a m_a a^{-1} \left(\delta_{(a)}^{(3)} u_{(a)}^i \right)_{,i} + \dot{\phi} \nabla^2 \delta\phi + \nabla \delta\phi \cdot \nabla \delta\dot{\phi} - \right. \\ \left. - \mathcal{H} \nabla^2 \delta F + \nabla^2 \delta \dot{F} - \dot{F} \nabla^2 \Phi + 2 \left(\frac{1}{\kappa} - F \right) \left(\mathcal{H} \nabla^2 \Phi + \nabla^2 \dot{\Psi} \right) \right] ; \end{aligned} \quad (18)$$

$$\begin{aligned} \frac{4}{3} \nabla^2 \Psi - \frac{1}{3} \nabla^2 \Phi = 8\pi G^* \left[\frac{1}{2} \left(\sum_a m_a a^{-1} \delta_{(a)}^{(3)} - a^2 \langle \rho_m \rangle \right) + \frac{1}{3} |\nabla \delta\phi|^2 + a^2 (V - \langle V \rangle) + \right. \\ \left. + \Phi \ddot{F}_b - \frac{1}{2} \ddot{F} - 2\mathcal{H} \delta \dot{F} + \dot{F} (4\mathcal{H} \Phi + \frac{1}{2} \dot{\Phi} + \frac{5}{2} \dot{\Psi}) + \frac{5}{6} \nabla^2 \delta F + \left(\frac{1}{\kappa} - F \right) \frac{\nabla^2}{3} (-\Phi + 4\Psi) \right] ; \end{aligned} \quad (19)$$

$$\begin{aligned} \delta \ddot{\phi} + 2\mathcal{H} \delta \dot{\phi} - (\dot{\Phi} + 3\dot{\Psi}) \dot{\phi}_b - \nabla^2 \delta\phi + a^2 [V'(\phi(\mathbf{x}, \eta)) - V'(\phi_b)] + \\ - \frac{a^2}{2} \delta F' R_b + (F'_b + \delta F') (\nabla^2 \Phi - 2\nabla^2 \Psi) = 0 ; \end{aligned} \quad (20)$$

$$\ddot{x}_a^i + \mathcal{H} \dot{x}_a^i = -\partial^i \Phi \quad (21)$$

In order to further simplify the equations, we have to estimate the characteristic evolution time for scalar field perturbations. We will follow the approach of [40], considering the scalar field perturbation as the sum of a “relativistic” perturbation $\delta\phi_R$ and a “non-relativistic” one, $\delta\phi_{NR}$,

$$\delta\phi(\mathbf{x}, \eta) \equiv \delta\phi_R(\mathbf{x}, \eta) + \delta\phi_{NR}(\mathbf{x}, \eta) \quad (22)$$

where $\delta\phi_{NR}$ is defined as the solution of the perturbed Klein-Gordon equation in the $c \rightarrow \infty$ limit. By definition, the time derivative of $\delta\phi_{NR}$ is negligible with respect to that of $\delta\phi_R$, which, on the contrary, has a wave-like behavior. The characteristic (conformal) time for the variation of $\delta\phi_{NR}$, τ_{NR} , is much larger than the characteristic time for the variation of $\delta\phi_R$, τ_R : on scales much smaller than the horizon, the time-variation of the non-relativistic component of scalar field perturbations is negligible with respect to the spatial gradient, which is equivalent to the limit $c \rightarrow \infty$. The Klein-Gordon equation for $\delta\phi_{NR}$ reduces to

$$-\nabla^2 \delta\phi_{NR} + a^2 \delta V'(\delta\phi_{NR}(\mathbf{x}, \eta)) + (F'_b + \delta F'_{NR}) \nabla^2 (\Phi - 2\Psi) - \frac{a^2}{2} \delta F'_{NR} R_b = 0 \quad (23)$$

where $\delta V'(\delta\phi_{NR}(\mathbf{x}, \eta)) \equiv V'(\phi_b(\eta) + \delta\phi_{NR}(\mathbf{x}, \eta)) - V'(\phi_b(\eta))$, and $\delta F_{NR} \equiv F(\phi_b + \delta\phi_{NR}) - F(\phi_b)$. Inserting the definition (22) into eq. (20), and using (23), we obtain the equation for $\delta\phi_R$:

$$\begin{aligned} \delta \ddot{\phi}_R + 2\mathcal{H} \delta \dot{\phi}_R - \nabla^2 \delta\phi_R + a^2 [V'(\phi_b + \delta\phi_R + \delta\phi_{NR}) - V'(\phi_b + \delta\phi_{NR})] - \frac{a^2}{2} [F'(\phi_b + \delta\phi_{NR} + \delta\phi_R) - \\ - F'(\phi_b + \delta\phi_{NR})] R_b + [F'(\phi_b + \delta\phi_{NR} + \delta\phi_R) - F'(\phi_b + \delta\phi_{NR})] \nabla^2 (\Phi - 2\Psi) = 0 . \end{aligned} \quad (24)$$

In the last equation we neglected all terms of order \mathcal{H}^2 : since $\delta\dot{\phi}_{NR}/\delta\phi_{NR} \sim \tau_{NR}^{-1}$, we have

$$\delta \ddot{\phi}_{NR}/\delta\phi_{NR} \sim \tau_{NR}^{-2} \lesssim \mathcal{H}^2; \quad 2\mathcal{H} \delta \dot{\phi}_{NR}/\delta\phi_{NR} \sim \tau_{NR}^{-1} \tau^{-1} \lesssim \mathcal{H}^2; \quad (\dot{\Phi} + 3\dot{\Psi}) \dot{\phi}_b \sim \mathcal{H}^2. \quad (25)$$

We assume that, on the scales characteristic for structure formation, $\delta\phi_R$ is a small perturbation (this assumption will be verified *a posteriori*); in such a case, $V'(\phi_b + \delta\phi_R + \delta\phi_{NR}) - V'(\phi_b + \delta\phi_{NR}) \sim V''(\phi_b + \delta\phi_{NR}) \delta\phi_R$, and $F'(\phi_b + \delta\phi_R + \delta\phi_{NR}) - F'(\phi_b + \delta\phi_{NR}) \sim F''(\phi_b + \delta\phi_{NR}) \delta\phi_R$. Substituting in eq. (24), there will be two terms containing $F''(\phi_b + \delta\phi_{NR}) \delta\phi_R$; they will produce, respectively, a term of order $\delta\phi_R k^2 (\Phi - 2\Psi)$ and a term of order $\delta\phi_R \mathcal{H}^2$, which are both negligible with respect to $\nabla^2 \delta\phi_R$. Therefore, eq. (24) becomes

$$\delta \ddot{\phi}_R + 2\mathcal{H} \delta \dot{\phi}_R - \nabla^2 \delta\phi_R + a^2 V''(\phi_b + \delta\phi_{NR}) \delta\phi_R = 0 . \quad (26)$$

This equation describes a (quasi-massless) plane-wave, whose amplitude is damped by the cosmic expansion, so that in a timescale $\tau \sim \mathcal{H}^{-1}$ the relativistic perturbation $\delta\phi_{\text{R}}$ vanishes on scales smaller than the horizon, $\delta\phi_{\text{R}} \rightarrow 0$. The mass associated to this wave is $a^2 V''(\phi_{\text{b}} + \delta\phi_{\text{NR}}) \sim \mathcal{H}^2$, negligible on sub-horizon scales: as the perturbation $\delta\phi_{\text{R}}$ enters the horizon, it will behave as radiation. We have thus verified *a posteriori* the validity of our assumption on the smallness of $\delta\phi_{\text{R}}$, which will be neglected hereafter. From now on, $\delta\phi$ has to be understood as $\delta\phi_{\text{NR}}$, the non-relativistic component of the scalar field perturbation, and we will omit the subscript NR ; this will be, in general, a non-linear perturbation.

In the Newtonian limit we can neglect time derivatives in the trace equation and in the energy constraint. Indeed, the scales which we are considering are well below the horizon, so the Laplacian terms will dominate over terms of order \mathcal{H}^2 . Note that $\dot{\phi}_{\text{b}}^2$ also is of order \mathcal{H}^2 , as a consequence of the Friedmann equations and of the smallness of the coupling at low redshifts; we will neglect $\dot{\phi}$ in this limit.

The energy constraint reads:

$$F\nabla^2\Phi = \left[\frac{1}{2} \left(\sum_a a^{-1} m_a \delta_{(a)}^{(3)} - a^2 \langle \rho_m \rangle \right) - a^2 (V - \langle V \rangle) - \frac{1}{2} \nabla^2 \delta F \right] \quad (27)$$

and the trace equation reads

$$\frac{F}{3} (4\nabla^2\Psi - \nabla^2\Phi) = \left[\frac{1}{2} \left(\sum_a m a^{-1} \delta_{(a)}^{(3)} - a^2 \langle \rho_m \rangle \right) + \frac{1}{3} |\nabla\delta\phi|^2 + a^2 (V - \langle V \rangle) + \frac{5}{6} \nabla^2 \delta F \right] \quad (28)$$

In the equations above, $\delta\phi \equiv \delta\phi_{\text{NR}}$. Note that, as anticipated in the previous section, the system (17)-(19) provides a redundant set of equations for the gravitational potentials Φ and Ψ : the (0-0) equation and the trace equation fully specify their evolution, so we can get rid of the spatial covariant derivative of the (0- i) Einstein's equations. Using eq. (27) and (28), together with eq. (23), we solve for $\Phi, \Psi, \delta\phi$:

$$\nabla^2\Psi = \frac{1}{2F} \left[a^2 \delta\rho_m + a^2 \delta V + \nabla^2 \delta F + \frac{1}{2} |\nabla\delta\phi|^2 \right] ; \quad (29)$$

$$\nabla^2\Phi = \frac{1}{2F} \left[a^2 \delta\rho_m - 2a^2 \delta V - \nabla^2 \delta F \right] ; \quad (30)$$

where $\delta V \equiv V(\phi_{\text{b}} + \delta\phi) - V(\phi_{\text{b}})$ and $a^2 \delta\rho_m \equiv \sum_a a^{-1} m_a \delta_{(a)}^{(3)} - a^2 \langle \rho_m \rangle$. The perturbed Klein-Gordon equation becomes

$$\nabla^2 \delta\phi = a^2 \delta V' - \frac{F'_{\text{b}} + \delta F'}{2F} \left[a^2 \delta\rho_m + 4a^2 \delta V + 3\nabla^2 \delta F + |\nabla\delta\phi|^2 \right] - \frac{a^2}{2} \delta F' R_{\text{b}} \quad (31)$$

Once the function F is specified together with the scalar field potential V , the system of equations (21),(30), (31) fully determines the evolution of perturbations in matter, scalar field and gravitational potentials; it can be numerically integrated, given the appropriate initial and boundary conditions. It is interesting to note that, for the “ordinary Quintessence” models, i.e. minimally coupled scalar field, the non-relativistic perturbations of the field vanish on sub-horizon scales as a consequence of the vanishing anisotropic stress (see Appendix B and C); however, this is generally not true for the extended models we are dealing with: the scalar field coupling to the Ricci scalar affects the final Poisson equation for the gravitational potentials, and can give rise to substantial modifications of the standard structure formation picture.

In the next section, we will consider the special case of non-minimal coupling analyzed in [21], adopting the current, local constraints on the value of the coupling function F .

IV. EXTENDED QUINTESSENCE

Now, let us focus on the “Extended Quintessence” model of [21]; in this model,

$$F = \kappa^{-1} + \zeta\phi^2 \quad (32)$$

where κ is a universal constant, whose unknown value is proportional to the “bare” gravitational constant G^* ; in scalar-tensor theories, G^* can in principle deviate from the Newtonian constant measured in Cavendish-type experiments [28]. Indeed, the actual Newtonian force between two close test masses measured at the present time in such experiments is proportional to

$$G_{eff}|_0 \equiv G = \frac{G^*}{\kappa F_0} \left(\frac{\omega_{JBD,0} + 2}{\omega_{JBD,0} + 3/2} \right) \sim \frac{G^*}{\kappa F_0} \equiv G_N . \quad (33)$$

In the equation above, $\omega_{JBD,0}$ is the present value of the Jordan-Brans-Dicke (JBD) parameter, which reduces to

$$\frac{1}{\omega_{JBD}} \equiv \frac{F_0'^2}{F_0} \quad (34)$$

when the kinetic factor $\omega(\phi)$ in the Lagrangian (2) is chosen to be a constant, as in our case.

The term between brackets in eq. (33) is due to the exchange of a scalar particle between the two test masses: since the JBD parameter is bound from solar-system experiments to be larger than ~ 3000 ([59], [60]), the present values of G_{eff} and G_N almost coincide. F is the coupling function entering in the Lagrangian of the theory, and, as we have seen, it enters in the Friedmann equations and in the Poisson equations for the local gravitational potentials; the value of this function at present time can be determined only *locally*, i.e. on the length scales where we are able to perform gravitational experiments to determine the “effective” gravitational constant. Thus, we can give a local representation of the function (32):

$$F = 8\pi G^{-1} + \zeta(\phi^2 - \phi_0^2) \equiv \frac{1}{8\pi G}(1 + y(\phi)) , \quad (35)$$

where the dimensionless function $y(\phi) \equiv 8\pi G\zeta(\phi^2 - \phi_0^2) \ll 1$, at low redshifts (see [29]). Thus, on the scales where the effective Newtonian constant reduces to the present value of G , (32) reduces to (35).

In this scheme, the present value of the coupling function reduces locally to the inverse of the measured Newtonian constant $F_0 = (8\pi G)^{-1}$.

We will expand equations (30) and (31) in terms of the perturbative parameter $y(\phi_0)$, where the subscript “0” refers to the present epoch. Note that

$$\frac{1}{F} = \frac{8\pi G}{(1 + y)} \sim 8\pi G(1 - y) \quad ;$$

for simplicity, we will assume that the scalar field potential perturbations δV are negligible: in order for this condition to hold, even for non-linear scalar field perturbations, the potential must be sufficiently flat. If this is the case, eq. (30) can be written as

$$\nabla^2 \Phi = 4\pi G(1 - y) \left[a^2 \delta \rho_m - \frac{\nabla^2 \delta y}{8\pi G} \right] \quad (36)$$

First of all, let us analyze the effect of the background scalar field. In the case of an unperturbed non-minimally coupled scalar field, the Poisson equation is modified by a factor $(1 - y)$ with respect to the standard case; this correction is scale-independent and proportional to the coupling parameter ζ at any redshift, reducing to zero at the present time. However, at redshifts relevant for the onset of structure formation, we expect the gravitational force

properties to be affected by this correction, which can be positive or negative depending on the sign of the coupling parameter ζ .

In order to appreciate any substantial correction to the standard Poisson equation which may be induced by the perturbations in the coupled scalar field, we have to evaluate under which conditions, and on which length scales L , if any, the two terms in square brackets in eq. (36) are comparable. Note that

$$4\pi G a^2 \delta \rho_m \sim \mathcal{H}^2 \frac{\delta \rho_m}{\rho_m} = L_H^{-2} \delta_m \quad ,$$

L_H being the Hubble length and δ_m the Dark Matter density contrast. Now, on the scales of clusters, i.e. length-scales $\sim 10^{-3} L_H$, where matter fluctuations are becoming non-linear today, in order for the two terms on the rhs of eq. (36) to be comparable, it would be sufficient to produce fluctuations $\delta y \sim 10^{-6}$. On galactic scales, overdensities are of order 10^4 , the typical scale being $10^{-5} L_H$, so again a perturbation $\delta y \sim 10^{-6}$ would affect at a considerable level the Poisson equation.

However, we are not free to establish the amount of fluctuations in the scalar field component: there are observational constraints, coming from the upper limits on the time variation of the gravitational constant and from solar-system limits on the JBD parameter ω_{JBD} [59]. The latter is the major constraint, since the Newtonian limit of the Klein-Gordon equation is not affected by time variations of the scalar field; we must have

$$\frac{1}{\omega_{JBD}} \equiv \frac{F_0'^2}{F_0} = 4\zeta^2 \phi_0^2 8\pi G < 2 \cdot 10^{-4} \quad (37)$$

where we selected a conservative lower limit $\omega_{JBD} > 5000 \equiv \omega_{lower}$. The constraint on ω_{JBD} is thus translated into an upper limit on the combination $\zeta \phi_0$, since $\sqrt{\omega_{JBD}^{-1}} = \pm 2\zeta \phi_0 \sqrt{8\pi G}$; for positive ζ , it must be

$$\zeta \phi_0 \sqrt{8\pi G} = \frac{1}{2} \sqrt{\omega_{JBD}^{-1}} \lesssim 7 \cdot 10^{-3} = \frac{1}{2} \sqrt{\omega_{lower}^{-1}} \quad (38)$$

which gives

$$F_0'/2F_0 = \frac{1}{2} \sqrt{8\pi G \omega_{JBD}^{-1}} = \zeta \phi_0 8\pi G \lesssim \sqrt{8\pi G} \cdot 7 \cdot 10^{-3} = \frac{\sqrt{8\pi G \omega_{lower}^{-1}}}{2} \quad (39)$$

It is important to emphasize that the condition above has to be thought as applying to the total (background plus perturbations) quantities F, F', ϕ at the present time; i.e. it does not represent a constraint on the present value of F_b'/F_b only, since this value could not be observationally discerned from possible local fluctuations; rather, the constraint (39) applies to the total ratio $F_0'/2F_0$, which appears in eq. (31). As we will see, this reflects into a constraint on $\zeta \phi_0$ rather than on $\zeta \phi_{b,0}$.

Hereafter, we will omit the subscript “0”, referring to the value at present (or at low-redshifts) of the quantities entering into the Klein-Gordon equation.

To check whether solar-system constraints on F'/F are compatible with the requirement $\delta y \sim 10^{-6}$, thus allowing for potentially observable imprints on the structure formation process, we found it convenient to separate the regime of linear $\delta \phi \cdot \sqrt{8\pi G} \ll 1$ from the more general case $\delta \phi \cdot \sqrt{8\pi G} \gtrsim 1$. Let us define the dimensionless scalar field variable $\tilde{\phi} \equiv \sqrt{8\pi G} \phi$. In the first case, inserting the upper limit (39) into eq. (31), we have

$$\nabla^2 \delta \tilde{\phi} \lesssim \frac{1}{2} \sqrt{\omega_{lower}^{-1}} \left(\delta_m \mathcal{H}^2 + 6\zeta \nabla^2 \delta \tilde{\phi} \right) \quad , \quad (40)$$

where the linearity of $\delta \phi$ allowed to neglect δF with respect to F_b in the factor in front of the brackets, as well as the term proportional to $|\delta \phi|^2$; the last term on the RHS of eq. (31) can be neglected on scales much smaller than the horizon. The quantities in eq. (40) have to be understood as their present values. It is evident that

$$\delta \tilde{\phi} \lesssim \frac{1}{2} \sqrt{\omega_{lower}^{-1}} \delta_m \left(\frac{L}{L_H} \right)^2 \quad ; \quad (41)$$

as we have seen, one has a typical value of $\delta_m (L/L_H)^2 \sim 10^{-6}$ on galaxy and cluster scales: therefore, the *linear* value of $\delta\phi\sqrt{8\pi G}$ on those scales is forced to be of the order of 10^{-8} (corresponding to $\delta y \sim 10^{-11}$), which is orders of magnitude below the value required for scalar field perturbations to affect the Poisson equation in a considerable way, so that the only effect on the Poisson equation would be given by the overall factor $(1 - y)$, which is active at low redshifts even in the absence of field perturbations.

To analyze the most general case, we will reject the linearity assumption for $\delta\phi$. On scales much smaller than the horizon, and neglecting the field potential, the Klein-Gordon equation (31) for the model (35) reduces in Fourier space to

$$\delta\tilde{\phi} = -\frac{\sqrt{\omega_{JBD}^{-1}}}{2} \left[\delta_m \left(\frac{L}{L_H} \right)^2 + 3\zeta\delta\tilde{\phi}(2\tilde{\phi} - \delta\tilde{\phi}) + \delta\tilde{\phi}^2 \right], \quad (42)$$

where we used the non-linear expression for $\delta F \equiv F - F_b = \zeta\delta\phi(\delta\phi + 2\phi_b) = \zeta\delta\phi(2\phi - \delta\phi)$.

Without loss of generality, we can assume that today $\tilde{\phi} = 1$, because we have only a constraint on the product $\zeta\tilde{\phi}$; with this choice, $\zeta < \omega_{lower}^{-1/2}/2 \sim 10^{-3}$ and eq. (42) becomes, dividing by $\sqrt{\omega_{JBD}^{-1}} \neq 0$,

$$\sqrt{\omega_{JBD}} \delta\tilde{\phi} + \delta\tilde{\phi}^2 + \delta_m \left(\frac{L}{L_H} \right)^2 \sim 0 \quad (43)$$

On galaxy and clusters scales, a typical value is $\delta_m \left(\frac{L}{L_H} \right)^2 \sim 10^{-6}$, much smaller than the lower limit on $\sqrt{\omega_{JBD}} \gtrsim \sqrt{\omega_{lower}} \sim 10^2$; the two solutions of (43) are therefore $\delta\tilde{\phi} \sim 0$ (which trivially corresponds to the solution previously found under the assumption of linearity), and $\delta\tilde{\phi} \sim -\sqrt{\omega_{JBD}} \sim 10^2$, which is a strongly non-linear scalar field perturbation; as for the Poisson equation (36), the term δy is, for $\delta\tilde{\phi} \sim 10^2$, of order 10^{-1} , much bigger than the value required for the two terms in brackets in eq. (36) to be comparable. The fact that such a value does not depend on the matter overdensity at any redshift, makes us argue that it is unphysical, and that the Extended Quintessence perturbations can only be linear as long as the constraint (37) applies.

However, we want to stress that an initially small scalar field perturbation could, in this model, grow and become non-linear on sub-horizon scales, since the limit (37) is only restricted to the Solar System neighborhood: no constraints on spatial fluctuations of the field are available on larger scales.

Note again that this kind of “growing” solution is only allowed in the non-minimally coupled case, since we assumed $\sqrt{\omega_{JBD}^{-1}} \neq 0$: the “Extended Quintessence” model could in principle admit substantial perturbations of the scalar field, as their time evolution, according to eq. (23), is sourced by a non-vanishing term which is ultimately related to the non-vanishing anisotropic stress (see Appendix B and C) and to the matter perturbations themselves.

V. CONCLUSIONS

In this paper we extended the Newtonian approximation to the class of scalar-tensor theories of gravity, where a non-minimally coupled scalar field is assumed to be responsible for the cosmic acceleration today. We obtained the equations relevant for simulations of gravitational clustering in these cosmological scenarios. The Newtonian Poisson equation acquires new contributions from scalar field perturbations, which may turn out to affect the gravitational collapse in an interesting way.

As already argued in a previous study of linear perturbations in these models [29], a substantial part of scalar field perturbations, along with perturbations in its stress-energy tensor components, is powered by perturbations in the matter component: this “gravitational dragging” is only due to the coupling of the scalar field with the Ricci scalar

(and, ultimately, with the perturbed matter), and can drive scalar field perturbations into the non-linear regime. On the contrary, self-interaction originated perturbations in the scalar field will unavoidably damp out on the scales relevant for structure formation, as in the case of minimally-coupled fields.

In order to get specific insights on the role of gravitational dragging, we specialized the final equations to the “Extended Quintessence” model of [21], where the non-minimal coupling strength is quantified by a dimensionless coupling parameter. Without adding a potential for the scalar field, we analyzed the result of the Poisson equation for the gravitational potentials and for the scalar field perturbations. We isolated and quantified the effect of the time-variation of the effective gravitational constant in the equation for clustering (36), produced by the time evolution of the background value of the scalar field. We also analyzed the effect of perturbations of the effective gravitational constant, finding that, in Extended Quintessence, the scalar field perturbations are prevented from growing non-linear by the Solar System constraints on the coupling parameter. Namely, structure formation is *locally* not affected by the modified gravity at a detectable level; in principle, however, scalar-field non linearities are not precluded even in the Extended Quintessence scenario, at least on those scales where the coupling parameter is allowed to vary over a less restrictive range of values. Further analysis is required in order to quantify the effects on forming structures in more general scalar-tensor theories.

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APPENDIX A: PERTURBATIONS IN ENERGY DENSITY AND PRESSURE

Let us decompose the scalar field stress-energy tensor as follows [29]:

$$T_j^{i(mc)} = \phi^i \phi_{,j} - \frac{1}{2} \delta_j^i (\phi_{,c} \phi^{,c} + 2V) \quad ; \quad T_j^{i(nmc)} = F_{;j}^i - \delta_j^i F_{;c}^c \quad ; \quad T_j^{i(grav)} = \left(\frac{1}{\kappa} - F \right) G_j^i . \quad (A1)$$

The background and perturbed energy densities and isotropic pressure, defined through $T_0^0 = -\rho$ and $T_i^i = p/3$, turn out to be

- Minimal coupling

$$\rho_{\phi_b}^{(mc)} = \frac{1}{2a^2} \dot{\phi}_b^2 + V(\phi_b) \quad ; \quad p_{\phi_b}^{(mc)} = \frac{1}{2a^2} \dot{\phi}_b^2 - V(\phi_b) \quad ; \quad (A2)$$

$$\delta\rho_{\phi}^{(mc)} = \frac{1}{2a^2} (1 - 2\Phi) (\delta\dot{\phi}^2 + 2\dot{\phi}_b \delta\dot{\phi}) + \frac{1}{2a^2} (1 + 2\Psi) |\nabla\delta\phi|^2 + \delta V - \Phi \frac{\dot{\phi}_b^2}{a^2} \quad (A3)$$

$$\delta p_{\phi}^{(mc)} = \frac{1}{2a^2} (1 - 2\Phi) (\delta\dot{\phi}^2 + 2\dot{\phi}_b \delta\dot{\phi}) - \frac{1}{6a^2} (1 + 2\Psi) |\nabla\delta\phi|^2 - \delta V - \Phi \frac{\dot{\phi}_b^2}{a^2} \quad (A4)$$

- Non-minimal coupling

$$\rho_{\phi_b}^{(nmc)} = -3 \frac{\mathcal{H}}{a^2} \dot{F}_b \quad ; \quad p_{\phi_b}^{(nmc)} = \frac{1}{a^2} \ddot{F}_b + \frac{\mathcal{H}}{a^2} \dot{F}_b \quad ; \quad (A5)$$

$$\delta\rho_{\phi}^{(nmc)} = \frac{1}{2a^2} [3\dot{F}_b (\dot{\Psi} + 2\mathcal{H}\Phi) + 3\delta\dot{F} (\dot{\Psi} + 2\mathcal{H}\Phi - \mathcal{H}) - \nabla\Psi\nabla\delta F + (1 + 2\Psi)\nabla^2\delta F] \quad (A6)$$

$$\delta p_\phi^{(nmc)} = -\frac{2}{a^2}\Phi\ddot{F}_b + \frac{1}{a^2}(1-2\Phi)\delta\ddot{F} - \frac{\dot{F}_b}{a^2}(\mathcal{H}-2\mathcal{H}\Phi-\dot{\Phi}-2\dot{\Psi}) + \frac{1}{3a^2}\nabla\delta F\nabla(2\Psi-3\Phi) - \frac{2}{3a^2}(1+2\Psi)\nabla^2\delta F \quad (\text{A7})$$

• Gravitational

$$\rho_{\phi_b}^{(grav)} = 3\frac{\mathcal{H}^2}{a^2}(\kappa^{-1}-F_b) \quad ; \quad p_{\phi_b}^{(nmc)} = -\frac{1}{a^2}(\kappa^{-1}-F_b)(2\dot{\mathcal{H}}+\mathcal{H}^2) \quad ; \quad (\text{A8})$$

$$\delta\rho_\phi^{(grav)} = -3\frac{\mathcal{H}^2}{a^2}\delta F + \frac{1}{a^2}(\kappa^{-1}-F_b-\delta F)(2\nabla^2\Psi-6\mathcal{H}^2\Phi-6\mathcal{H}\dot{\Psi}) \quad (\text{A9})$$

$$\delta p_\phi^{(grav)} = (2\dot{\mathcal{H}}+\mathcal{H}^2)\frac{\delta F}{a^2} + \frac{2}{a^2}(\kappa^{-1}-F_b-\delta F)\left[(2\dot{\mathcal{H}}+\mathcal{H}^2)\Phi+\mathcal{H}\dot{\Phi}+\ddot{\Psi}+2\mathcal{H}\dot{\Psi}+\frac{1}{3}\nabla^2(\Phi-\Psi)\right] \quad (\text{A10})$$

In the equations above, the subscript b refers to background quantities.

In the Newtonian limit, we have

$$(\delta\rho_\phi+3\delta p_\phi)^{(mc)} \rightarrow -2\delta V \quad (\text{A11})$$

$$(\delta\rho_\phi+3\delta p_\phi)^{(nmc)} \rightarrow -\frac{\nabla^2\delta F}{a^2} \quad (\text{A12})$$

$$(\delta\rho_\phi+3\delta p_\phi)^{(grav)} \rightarrow (\kappa^{-1}-F)\frac{2\nabla^2\Phi}{a^2} \quad . \quad (\text{A13})$$

At the same time, it is easy to verify that the Newtonian limit of the perturbed Einstein equation $\delta R_0^0 = \kappa\delta S_0^0$ gives

$$\nabla^2\Phi = \frac{a^2}{2}\kappa(\delta\rho+3\delta p) \quad (\text{A14})$$

where $\delta\rho$ and δp include contributions from matter and from scalar field (minimal coupling, non-minimal coupling and gravitational parts). Substituting the limits (A11)-(A13) into (A14), and comparing with eq. (30) we see that the contribution $-2a^2\delta V$ in eq. (30) originates from the minimal coupling, the term $-\nabla^2\delta F$ originates from the non-minimal coupling, while the gravitational contribution can be absorbed in the quantity F^{-1} . Looking at eq. (30), we see that the role of the non-minimal coupling is twofold: on the one hand, it generates the energy and pressure contribution $-\nabla^2\delta F$, which is non-vanishing only if the scalar field fluctuations are non vanishing; on the other hand, it also generates a modification of the gravitational constant, where the effective gravitational constant F^{-1} is generally different from κ even if the scalar field does not have fluctuations.

APPENDIX B: ANISOTROPIC STRESS

The unperturbed space-space component of the Einstein tensor has only trace component:

$$G_{j_b}^i = -\frac{2}{a^2}(\dot{\mathcal{H}}+\frac{1}{2}\mathcal{H}^2)\delta_j^i \quad (\text{B1})$$

On the other hand, the perturbations δG_j^i have a trace and a traceless part:

$$\delta G_{j(trace)}^i = \frac{2}{a^2}\delta_j^i\left[(2\dot{\mathcal{H}}+\mathcal{H}^2)\Phi+\mathcal{H}\dot{\Phi}+\ddot{\Psi}+2\mathcal{H}\dot{\Psi}+\frac{1}{3}(\nabla^2\Phi-\nabla^2\Psi)\right] \quad (\text{B2})$$

and

$$\delta G_{j(traceless)}^i = \frac{1}{a^2} \left[-\gamma^{ik}(\Phi - \Psi)_{,jk} + \frac{1}{3} \nabla^2(\Phi - \Psi) \delta_j^i \right] + (vect. \text{ and tensor pert.}) \quad (B3)$$

Here, we allowed for tensor and vector perturbations, which we do not write explicitly (for details, see [62]).

The total stress energy tensor for the scalar field in the Extended Quintessence model can be decomposed as

$$T_\nu^\mu = T_\nu^{\mu(mc)} + T_\nu^{\mu(nmc)} + T_\nu^{\mu(grav)}$$

The anisotropic stress is defined in terms of the spatial components of the stress-energy tensor; the latter can always be decomposed as the sum of a trace tensor and a traceless tensor:

$$T_j^i \equiv p_b[\delta_j^i + \pi_L \delta_j^i + \pi_{Tj}^i], \quad (B4)$$

where p_b is the fluid pressure in the unperturbed state; π_L is the isotropic pressure perturbation (a commonly used definition is $\pi_L \equiv \delta p/p_b$), and $p_b \pi_{Tj}^i$ is the anisotropic stress tensor, defined as the traceless part of T_j^i :

$$p_b \pi_{Tj}^i \equiv T_j^i - \frac{1}{3} \delta_j^i T_k^k.$$

We are working in configuration space, so π_L and π_{Tj}^i are functions of (\mathbf{x}, η) .

Note that in linear theory we can write

$$\pi_{Tj}^i = (\gamma^{ik} \partial_i \partial_k - \frac{\delta_j^i}{3} \nabla^2) \pi_T$$

π_T being the amplitude of the anisotropic stress perturbation, see eg. [61], [62]).

In the following, we will consider linear perturbations in the metric (described in the conformal Newtonian gauge), but the scalar field perturbations will generally be non linear.

For the minimally coupled field, with $\omega = 1$, the tensor $T_j^{i(mc)}$ is given by

$$T_j^{i(mc)} = \phi^{,i} \phi_{,j} - \frac{1}{2} \delta_j^i (\phi_{,c} \phi^{,c} + 2V) = T_{j(trace)}^{i(mc)} + T_{j(traceless)}^{i(mc)}$$

where

$$T_{j(trace)}^{i(mc)} = \frac{1}{3a^2} (1 + 2\Psi) |\nabla \delta \phi|^2 \delta_j^i - \frac{1}{2} \delta_j^i (\phi_{,c} \phi^{,c} + 2V)$$

and

$$T_{j(traceless)}^{i(mc)} = \frac{1}{a^2} (1 + 2\Psi) \gamma^{ik} \partial_k \delta \phi \partial_j \delta \phi - \frac{1}{3a^2} (1 + 2\Psi) |\nabla \delta \phi|^2 \delta_j^i$$

Note that, since ϕ_b does not depend on the spatial coordinates, $\partial_i \phi = \partial_i \delta \phi$; for this reason, in the linear regime (i.e., for linear perturbations of the scalar field), the anisotropic stress from a minimally coupled scalar field is negligible, being second order in the scalar field perturbations. In general, the anisotropic stress from a minimally coupled scalar field is of order $\mathcal{O}(|\nabla \delta \phi|^2)$.

For the non-minimal coupling tensor,

$$T_j^{i(nmc)} = F_{;j}^{,i} - \delta_j^i F_{;c}^{,c} = T_{j(trace)}^{i(nmc)} + T_{j(traceless)}^{i(nmc)}$$

with

$$T_{j(trace)}^{i(nmc)} = -\frac{1}{3a^2} \delta_j^i \left[\nabla \delta F \nabla \Psi - (1 + 2\Psi) \nabla^2 \delta F + \dot{F} (3\mathcal{H} - 3\dot{\Psi} - 6\mathcal{H}\Phi) \right] - \delta_j^i F^{,c}_{;c}$$

and, after some algebra, it can be shown that

$$\begin{aligned} T_{j(traceless)}^i(nmc) &= \frac{1}{a^2}(1+2\Psi) \left[\gamma^{ik} F_{,kj} - \frac{1}{3} \delta_j^i \nabla^2 F \right] + \frac{1}{a^2} \gamma^{ik} \Psi_{,j} F_{,k} + \gamma^{i\lambda} \Psi_{,\lambda} \gamma_{kj} F^{,k} - \frac{2}{3a^2} \delta_j^i \nabla F \nabla \Psi = \\ &= \frac{(1+2\Psi)}{a^2} \left[\gamma^{ik} \delta F_{,kj} - \frac{1}{3} \delta_j^i \nabla^2 \delta F \right] + \frac{2}{a^2} \left[\gamma^{ik} \Psi_{,j} \delta F_{,k} - \frac{1}{3} \delta_j^i \nabla \delta F \nabla \Psi \right] \end{aligned}$$

We see from here that the anisotropic stress from the non-minimally coupled field contains terms which are proportional to the spatial gradients of F ; in general, these terms are of the order $\mathcal{O}(|\nabla \delta \phi|)$, so the anisotropic stress for a non-minimally coupled scalar field would survive even in the case of linear perturbations of the field.

Finally, the gravitational term is

$$T_j^{i(grav)} = \left(\frac{1}{\kappa} - F \right) G_j^i = T_{j(trace)}^{i(grav)} + T_{j(traceless)}^{i(grav)}$$

Using equations (B1), (B2) and (B3), we have

$$T_{j(trace)}^{i(grav)} = \left(\frac{1}{\kappa} - F \right) \frac{2}{a^2} \delta_j^i \left[-(\dot{\mathcal{H}} + \frac{1}{2} \mathcal{H}^2) + (2\dot{\mathcal{H}} + \mathcal{H}^2) \Phi + \mathcal{H} \dot{\Phi} + \ddot{\Psi} + 2\mathcal{H} \dot{\Psi} + \frac{1}{3} (\nabla^2 \Phi - \nabla^2 \Psi) \right]$$

and

$$T_{j(traceless)}^{i(grav)} = \left(\frac{1}{\kappa} - F \right) \frac{1}{a^2} \left[-\gamma^{ik} (\Phi - \Psi)_{,jk} + \frac{1}{3} \nabla^2 (\Phi - \Psi) \delta_j^i \right] + (vect. \text{ and tensor pert.})$$

Therefore, the total anisotropic stress, without taking into account vector and tensor perturbations of the metric, will be given by the sum of the three contributions from the minimal coupling, the non-minimal coupling and the gravitational terms. Since its background value is zero, we have

$$\begin{aligned} \delta T_{j(traceless)}^{i(mc+nmc+grav)} &= \frac{1}{a^2} (1+2\Psi) \gamma^{ik} \partial_k \delta \phi \partial_j \delta \phi - \frac{1}{3a^2} (1+2\Psi) |\nabla \delta \phi|^2 \delta_j^i + \\ &+ \frac{(1+2\Psi)}{a^2} \left[\gamma^{ik} \delta F_{,kj} - \frac{1}{3} \delta_j^i \nabla^2 \delta F \right] + \frac{2}{a^2} \left[\gamma^{ik} \Psi_{,j} \delta F_{,k} - \frac{1}{3} \delta_j^i \nabla \delta F \nabla \Psi \right] + \\ &+ \left(\frac{1}{\kappa} - F \right) \frac{1}{a^2} \left[-\gamma^{ik} (\Phi - \Psi)_{,jk} + \frac{1}{3} \nabla^2 (\Phi - \Psi) \delta_j^i \right] + (vect. \text{ and tensor pert.}) \end{aligned}$$

Using eq. (B3), and writing the gravitational contribution on the left-hand side of the traceless part of the perturbed space-space Einstein equation

$$\delta G_{j(traceless)}^i = \kappa \delta T_{j(traceless)}^i$$

we obtain:

$$\begin{aligned} F \left[-\gamma^{ik} (\Phi - \Psi)_{,jk} + \frac{1}{3} \nabla^2 (\Phi - \Psi) \delta_j^i \right] &= (1+2\Psi) \gamma^{ik} \partial_k \delta \phi \partial_j \delta \phi - \frac{1}{3a^2} (1+2\Psi) |\nabla \delta \phi|^2 \delta_j^i + \\ + \frac{(1+2\Psi)}{a^2} \left[\gamma^{ik} \delta F_{,kj} - \frac{1}{3} \delta_j^i \nabla^2 \delta F \right] &+ \frac{2}{a^2} \left[\gamma^{ik} \Psi_{,j} \delta F_{,k} - \frac{1}{3} \delta_j^i \nabla \delta F \nabla \Psi \right] , \end{aligned} \quad (B5)$$

where contributions from vectors and tensors have been omitted.

As an important consequence, we can note that, in the case of a minimally coupled scalar field (i.e., $F \equiv \kappa^{-1}$), one has

$$\Phi = \Psi + \mathcal{O}(\delta \phi)^2 ;$$

if $\delta \phi$ are linear perturbations in the minimally coupled scalar field, the difference between the two gravitational potentials is negligible in a first-order theory.

APPENDIX C: NEWTONIAN APPROXIMATION FOR THE MINIMALLY COUPLED SCALAR FIELD IN LINEAR PERTURBATION THEORY

In order to better understand the important role of the non-minimal coupling, let us see what to expect to be the perturbation behavior in the case of a minimally coupled scalar field.

First of all, in the minimally coupled case, the anisotropic pressure perturbation is of order $\mathcal{O}(\delta\phi)^2$, constraining the two gravitational potentials Φ and Ψ to differ by terms of the same order. Thus, we expect that the (Newtonian) Poisson equations (29), (30) will be equivalent in linear theory, for a minimally coupled scalar field.

Taking the trace of the perturbed space-space Einstein equation $\delta G_i^i = \kappa \delta T_i^i$ we obtain, for the case of minimally-coupled scalar field ($\kappa \equiv 8\pi G_N$),

$$(2\dot{\mathcal{H}} + \mathcal{H}^2)\Phi + \mathcal{H}\dot{\Phi} + \ddot{\Psi} + 2\mathcal{H}\dot{\Psi} + \frac{1}{3}(\nabla^2\Phi - \nabla^2\Psi) = \frac{\kappa}{2} \left[\frac{1}{2}(1 - 2\Phi)(\delta\dot{\phi}^2 + 2\dot{\phi}_b\delta\dot{\phi}) - \Phi\dot{\phi}_b^2 - \frac{1}{6}|\nabla\delta\phi|^2 - a^2\delta V \right] . \quad (\text{C1})$$

Here $\nabla^2(\Phi - \Psi) \sim \mathcal{O}|\nabla\delta\phi|^2$ can be neglected for linear perturbation of the field. As a consequence, $a^2\delta V$ will be of order \mathcal{H}^2 (in eq. (C1), $\delta\phi$ denotes non-relativistic perturbations); by differentiating $a^2\delta V$ with respect to the conformal time, one can show that $a^2\delta V' \sim \mathcal{O}(\mathcal{H}^2)$: inserting this result in the Newtonian limit of the perturbed Klein-Gordon equation for the non-relativistic scalar field perturbations (23), we can see that, for linear perturbations $\delta\phi$,

$$\nabla^2\delta\phi \sim a^2\delta V' \sim \mathcal{O}(\mathcal{H}^2) ,$$

which is negligible in the Newtonian limit of scales smaller than the horizon ; thus, the Klein-Gordon equations implies that the (minimally coupled) scalar field perturbations, in linear theory, will be negligible on those scales. For this reason, the scalar field will behave as a homogeneous component on the scales relevant for structure formation. Correspondingly, $\Phi \rightarrow \Psi$, and the two equations (30) and (29), with $F = (8\pi G^*)^{-1}$, will be identical (up to terms of order $(\mathcal{H}^2, \delta\phi^2)$).

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- [1] S. Perlmutter et al., Nature **391**, 51 (1998).
 - [2] A.G. Riess et al., Astron. J. **116**, 1009 (1998).
 - [3] D.N.Spergel et al., ApJ in press (2003).
 - [4] C.L. Bennett et al., ApJ in press (2003).
 - [5] S. Dodelson et al., ApJ **572**, 140 (2002).
 - [6] W.J. Percival et al., MNRAS **337**, 1068 (2002).
 - [7] L. Verde et al. MNRAS **335**, 432 (2002).
 - [8] W.L. Freedman et al., ApJ **553**, 47 (2001).
 - [9] B. Ratra and P.J.E. Peebles, Phys. Rev. D **37**, 3406 (1988).
 - [10] P.J.E. Peebles and B. Ratra, ApJ Lett. **325**, L17 (1988).
 - [11] K. Coble, S. Dodelson, and J.A. Frieman, Phys.Rev. D **55**, 1851 (1997).
 - [12] J.A. Frieman and I. Waga, Phys. Rev. D **57**, 4642 (1998).
 - [13] P.T.P. Viana and A.R. Liddle, Phys. Rev. D **57**, 674 (1998).
 - [14] P.G. Ferreira and M. Joyce, Phys. Rev. D **58**, 023503 (1998).
 - [15] E.J. Copeland, A.R. Liddle, and D. Wands, Phys. Rev. D **57**, 4686 (1998).
 - [16] R.R. Caldwell, R. Dave, and P.J. Steinhardt, Phys. Rev. Lett. **80**, 1582 (1998).
 - [17] I. Zlatev, L. Wang, and P.J. Steinhardt, Phys. Rev. Lett. **82**, 896 (1999).
 - [18] P.J. Steinhardt, L. Wang, and I. Zlatev, Phys. Rev. D **59**, 123504 (1999).
 - [19] L. Wang and P.J. Steinhardt, ApJ **508**, 483 (1998).
 - [20] T. Chiba, Phys.Rev. D **60**, 083508 (1999).
 - [21] F. Perrotta, C. Baccigalupi, and S. Matarrese, Phys. Rev. D **61**, 023507 (2000).
 - [22] L. Amendola, Phys. Rev. D **60**, 043501 (1999).
 - [23] L. Amendola, Phys. Rev. D **62**, 043511 (2000).

- [24] J.P. Uzan, Phys. Rev. D **59**, 123510 (1999).
- [25] N. Bartolo and M. Pietroni, Phys. Rev. D **61**, 023518 (2000).
- [26] O. Bertolami and P.J. Martins, Phys. Rev. D **61**, 064007 (2000).
- [27] C. Baccigalupi, S. Matarrese, and F. Perrotta, Phys. Rev. D **62**, 123510 (2000).
- [28] G. Esposito-Farese and D. Polarski, Phys. Rev. D **63**, 063504 (2001).
- [29] F. Perrotta and C. Baccigalupi, Phys. Rev. D **65**, 123505 (2002).
- [30] T. Damour, I.I. Kogan, and A. Papazoglou, Phys. Rev. D **66**, 104025 (2002).
- [31] S.M. Carroll, V. Duvvuri, M. Trodden, and M.S. Turner, astro-ph/0307018 (2003).
- [32] G. Dvali and M.S. Turner, astro-ph/0301510 (2003).
- [33] P. Caresia, S. Matarrese, and Moscardini L., astro-ph/0308147 (2003).
- [34] C. Wetterich, Phys. Lett. B **522**, 5 (2001).
- [35] C. Wetterich, Phys. Rev. D **65**, 123512 (2002).
- [36] N. Bilic, G.B. Tupper, and R.D. Viollier, Phys. Lett. B **535**, 17 (2002).
- [37] T. Matos, F.S. Guzman, and D. Nunez, Phys. Rev. D **62**, 061301 (2000).
- [38] C. Armendariz-Picon, V. Mukhanov, and P.J. Steinhardt, Phys. Rev. D **63**, 103510 (2001).
- [39] A. Arbey, J. Lesgourgues, and P. Salati, Phys. Rev. D **68**, 023511 (2003).
- [40] S. Matarrese, M. Pietroni, and C. Schmid, JCAP 0308 005 (2003).
- [41] M.B. Hoffman, astro-ph/0307350 (2003).
- [42] G.R. Farrar and P.J.E. Peebles, astro-ph/0307316 (2003).
- [43] P. Brax, J. Martin, A. Riazuelo, Phys. Rev. D **62**, 103505 (2000).
- [44] R. Bean, A. Melchiorri, Phys. Rev. D **65**, 041302 (2002).
- [45] C. Baccigalupi, A. Balbi, S. Matarrese, F. Perrotta, and N. Vittorio, Phys. Rev. D **65**, 063520 (2002).
- [46] A. Balbi, C. Baccigalupi, F. Perrotta, S. Matarrese, and N. Vittorio, ApJ **588**, L5 (2003).
- [47] P. Mukherjee, A.J. Banday, A. Riazuelo, K.M. Górski, and B. Ratra, astro-ph/0306147 (2003).
- [48] B. Boisseau, G. Esposito-Farese, D. Polarski, and A.A. Starobinsky, Phys. Rev. Lett. **85**, 2236 (2000).
- [49] W. Hu, ApJ **506**, 485 (1998).
- [50] C.P. Ma, R.R. Caldwell, P. Bode, and L. Wang, ApJ **521**, L1 (1999).
- [51] M. Bartelmann, M. Meneghetti, F. Perrotta, C. Baccigalupi, and L. Moscardini, A&A **409**, 449 (2003).
- [52] R. Mainini, L.P.L. Colombo, and S.A. Bonometto, New Astron. **8**, 751 (2003).
- [53] R. Mainini, A.V. Maccio', S.A. Bonometto, and A. Klypin, astro-ph/0303303 (2003).
- [54] A. Klypin, A.V. Maccio', R. Mainini, and S.A. Bonometto, astro-ph/0303304 (2003).
- [55] A.V. Maccio', C. Quercellini, R. Mainini, L. Amendola, and S.A. Bonometto, astro-ph/0309671 (2003).
- [56] E.V. Linder and A. Jenkins, astro-ph/0305286 (2003).
- [57] K. Dolag, M. Bartelmann, F. Perrotta, C. Baccigalupi, L. Moscardini, M. Meneghetti, and G. Tormen, astro-ph/0309771 (2003).
- [58] S. Weinberg, *Gravitation and Cosmology*, J. Wiley & Sons Ed., New York (1972).
- [59] C. Will, gr-qc/0103036.
- [60] T.M. Eubanks et al., Bull. Am. Phys. Soc., Abstract # K11.05 (1997).
- [61] C.P. Ma and E. Bertschinger, ApJ **455**, 7 (1995).
- [62] H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. **78**, 1 (1984).